Homework #2 – Solutions

Reading

- Chapter 2.1-2.5

Problems

1. (2.2) Electrical Conduction

The resistivity of aluminum at 25 °C has been measured to be $2.72 \times 10^{-8}\ \Omega$ m. The thermal coefficient of resistivity of aluminum at 0 °C is $4.29 \times 10^{-3}\ \text{K}^{-1}$. Aluminum has a valency of 3, a density of $2.70\ \text{g cm}^{-3}$, and an atomic mass of 27.

a. Calculate the resistivity of aluminum at -40 ºC.

b. What is the thermal coefficient of resistivity at -40 ºC?

c. Estimate the mean free time between collisions for the conduction electrons in aluminum at 25 °C, and hence estimate their drift mobility.

d. If the mean speed of the conduction electrons is about 2×10^6 m s^{-1}, calculate the mean free path and compare this with the size of a unit cell in Al (Al is FCC with a lattice parameter of 0.40478 nm). What should be the thickness of an Al film that is deposited on an IC chip such that its resistivity is the same as that of bulk Al?

e. What is the percentage change in the power loss due to Joule heating of the aluminum wire when the temperature drops from 25 °C to -40 ºC?

Solution

a. Apply the equation for temperature dependence of resistivity, $\rho(T) = \rho_o[1 + \alpha_o(T - T_o)]$. We have the temperature coefficient of resistivity, α_o, at T_o where T_o is the reference temperature. We can either work in K or °C inasmuch as only temperature changes are involved. The two given reference temperatures are 0 °C or 25 °C, depending on choice. Taking $T_o = 0$ °C,

$$\rho(-40°\text{C}) = \rho_o[1 + \alpha_o(-40°\text{C} - 0°\text{C})]$$

$$\rho(25°\text{C}) = \rho_o[1 + \alpha_o(25°\text{C} - 0°\text{C})]$$

Divide the above two equations to eliminate ρ_o,

$$\rho(-40°\text{C})/\rho(25°\text{C}) = [1 + \alpha_o(-40°\text{C})] / [1 + \alpha_o(25°\text{C})]$$

Next, substitute the given values $\rho(25°\text{C}) = 2.72 \times 10^{-8}\ \Omega$ m and $\alpha_o = 4.29 \times 10^{-3}\ \text{K}^{-1}$ to obtain

$$\rho(-40\ °\text{C}) = (2.72\times10^{-8}\ \Omega\ \text{m})\frac{[1+(4.29\times10^{-3})(-40)]}{[1+(4.29\times10^{-3})(25)]} = \mathbf{2.035 \times 10^{-8}\ \Omega\ m}$$

b. In $\rho(T) = \rho_o[1 + \alpha_o(T - T_o)]$ we have α_o at T_o where T_o is the reference temperature, for example, 0° C or 25 °C depending on choice. We will choose T_o to be first at 0 °C and then at –40 °C (= T_2) so that the resistivity at T_2 and then at T_o are:

At T_2, $\rho_2 = \rho_o[1 + \alpha_o(T_2 - T_o)]$; the reference being T_o and ρ_o which defines α_o

and at T_o $\rho_o = \rho_2[1 + \alpha_2(T_o - T_2)]$; the reference being T_2 and ρ_2 which defines α_2

Rearranging the above two equations we find

$$\alpha_2 = \alpha_o / [1 + (T_2 - T_o)\alpha_o]$$

i.e. $\boldsymbol{\alpha_{-40}} = (4.29 \times 10^{-3}) / [1 + (-40 - 0)(4.29 \times 10^{-3})] = \mathbf{5.18 \times 10^{-3}\ °C^{-1}}$

<u>Alternatively</u>, consider the definition of α_2 that is α_{-40}

From $$\alpha_o = \frac{1}{\rho_o}\left[\frac{d\rho}{dT}\right]_{T_o}$$

we have $\alpha_{-40} = \{1/[\rho(-40\ °C)]\} \times \{[\rho(25°C) - \rho(-40°C)] / [(25°C) - \rho(-40°C)]\}$

∴ $\alpha_{-40} = 1 / [(2.035 \times 10^{-8})] \times \{(2.72 \times 10^{-8}) - (2.035 \times 10^{-8})] / [(25) - (-40)]\}$

∴ $\boldsymbol{\alpha_{-40}} = \mathbf{5.18 \times 10^{-3}\ K^{-1}}$

c. We know that $1/\rho = \sigma = en\mu$ where σ is the electrical conductivity, e is the electron charge, and μ is the electron drift mobility. We also know that $\mu = e\tau / m_e$, where τ is the mean free time between electron collisions and m_e is the electron mass. Therefore,

$$1/\rho = e^2 n\tau/m_e$$

∴ $$\tau = m_e/\rho e^2 n \qquad (1)$$

Here n is the number of conduction electrons per unit volume. But, from the density d and atomic mass M_{at}, atomic concentration of Al is

$$n_{\text{Al}} = \frac{N_A d}{M_{\text{at}}} = \frac{(6.022 \times 10^{23}\ \text{mol}^{-1})(2700\ \text{kg/m}^3)}{(0.027\ \text{kg/mol})} = 6.022 \times 10^{28}\ \text{m}^{-3}$$

so that $n = 3n_{\text{Al}} = 1.807 \times 10^{29}\ \text{m}^{-3}$

assuming that each Al atom contributes 3 "free" conduction electrons to the metal and substituting into (1),

$$\tau = \frac{m_e}{\rho e^2 n} = \frac{(9.109 \times 10^{-31}\ \text{kg})}{(2.72 \times 10^{-8}\ \Omega\,\text{m})(1.602 \times 10^{-19}\ \text{C})^2(1.807 \times 10^{29}\ \text{m}^{-3})}$$

∴ $\boldsymbol{\tau} = \mathbf{7.22 \times 10^{-15}\ s}$

(Note: If you do not convert to meters and instead use centimeters you will not get the correct answer because seconds is an SI unit.)

The relation between the drift mobility μ_d and the mean free time is given by Equation 2.5, so that

$$\mu_d = \frac{e\tau}{m_e} = \frac{\left(1.602\times10^{-19}\text{C}\right)\left(7.22\times10^{-15}\text{s}\right)}{\left(9.109\times10^{-31}\text{kg}\right)}$$

$\therefore$ $\boldsymbol{\mu_d = 1.27 \times 10^{-3}\ \mathrm{m^2\ V^{-1}s^{-1}} = 12.7\ \mathrm{cm^2\ V^{-1}s^{-1}}}$

d. The mean free path is $l = u\tau$, where u is the mean speed. With $u \approx 2 \times 10^6$ m s^{-1} we find the mean free path:

$$l = u\tau = (2 \times 10^6\ \mathrm{m\ s^{-1}})(7.22 \times 10^{-15}\ \mathrm{s}) \approx \mathbf{1.44 \times 10^{-8}\ m \approx 14.4\ nm}$$

The lattice parameter of Al is 0.40478 nm. Thus $l/a = 35.6$. The electron traveling along the cube edge travels for about 35 unit cells before it is scattered.

A thin film of Al must have a much greater thickness than l to show bulk behavior (e.g., thickness $> 3l$ or $> \sim50$ nm). Otherwise, scattering from the surfaces will increase the resistivity by virtue of Matthiessen's rule.

e. Power $P = I^2R$ and is proportional to the resistivity ρ, assuming the rms current level stays relatively constant. Then we have

$$[P(-40\ °\mathrm{C}) - P(25\ °\mathrm{C})] / P(25\ °\mathrm{C}) = P(-40\ °\mathrm{C}) / P(25\ °\mathrm{C}) - 1 = \rho(-40\ °\mathrm{C}) / \rho(25\ °\mathrm{C}) - 1$$

$$= (2.03 \times 10^{-8}\ \Omega\ \mathrm{m} / 2.72 \times 10^{-8}\ \Omega\mathrm{m}) - 1 = \mathbf{-0.254}, \text{ or } \mathbf{-25.4\%}$$

(Negative sign means a reduction in the power loss).

2. (2.5) **Effective number of conduction electrons per atom**

a. Electron drift mobility in tin (Sn) is 3.9 cm^2 V^{-1} s^{-1}. The room temperature (20 °C) resistivity of Sn is about 110 nΩ m. Atomic mass M_{at} and density of Sn are 118.69 g mol^{-1} and 7.30 g cm^{-3}, respectively. How many “free” electrons are donated by each Sn atom in the crystal? How does this compare with the position of Sn in Group IVB of the Periodic Table?

b. Consider the resistivity of few selected metals from Groups I to IV in the Periodic Table in Table 2.8. Calculate the number of conduction electrons contributed per atom and compare this with the location of the element in the Periodic Table. What is your conclusion?

Table 2.8 Selection of metals from Groups I to IV in the Periodic Table

Metal	Periodic Group	Valency	Density (g cm^{-3})	Resistivity (nΩ m)	Mobility (cm^2 V^{-1}s^{-1})
Na	IA	1	0.97	42.0	53
Mg	IIA	2	1.74	44.5	17
Ag	IB	1	10.5	15.9	56
Zn	IIB	2	7.14	59.2	8
Al	IIIB	3	2.7	26.5	12
Sn	IVB	4	7.30	110	3.9
Pb	IVB	4	11.4	206	2.3

NOTE: Mobility from Hall-effect measurements.

Solution

a. Electron concentration can be calculated from the conductivity of Sn, $\sigma = en\mu_d$.

$$n_e = \frac{\sigma}{e\mu_d} = \frac{(110\times10^{-9}\,\Omega\,\mathrm{m})^{-1}}{(1.602\times10^{-19}\,\mathrm{C})(3.9\times10^{-4}\,\mathrm{m^2\,V^{-1}\,s^{-1}})}$$

$= 1.46\times10^{29}$ electrons m^3.

The atomic concentration, *i.e.* number of Sn atoms per unit volume is

$$n_{\mathrm{at}} = \frac{dN_A}{M_{\mathrm{at}}} = \frac{(7.3\times10^3\,\mathrm{kg})(6.022\times10^{23}\,\mathrm{mol^{-1}})}{(118.69\times10^{-3}\,\mathrm{kg\,mol^{-1}})}$$

$= 3.70 \times 10^{28}$ Sn atoms m^{-3}.

Hence the number of electrons donated by each atom is $(n_e/n_{\mathrm{at}}) = 3.94$ or **4** electrons per Sn atom. This is in good agreement with the position of the Sn in the Periodic Table (IVB) and its valency of 4.

b. Using the same method used above, the number of electrons donated by each atom of the element are calculated and tabulated as follows in Table 2Q05.

Table 2Q05 Number of electrons donated by various elements

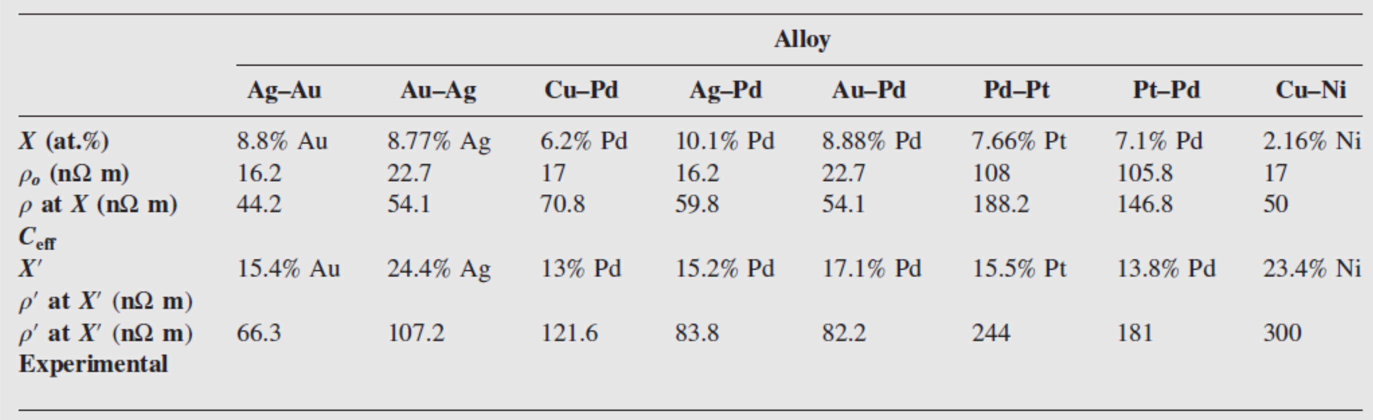
Metal	Periodic Group	Valency	Atomic Concentration n_{at} (m^{-3})	Electron Concentration n_e (m^{-3})	Number of electrons n_e/n_{at}	Integer (n_e/n_{at})
Na	IA	1	2.541×10^{28}	2.808×10^{28}	1.11	1
Mg	IIA	2	4.311×10^{28}	8.262×10^{28}	1.91	2
Ag	IB	1	5.862×10^{28}	7.019×10^{28}	1.20	1
Zn	IIB	2	6.575×10^{28}	1.320×10^{29}	2.00	2
Al	IIIB	3	6.026×10^{28}	1.965×10^{29}	3.26	3
Sn	IVB	4	3.703×10^{28}	1.457×10^{29}	3.93	4
Pb	IVB	4	3.313×10^{28}	1.319×10^{29}	3.98	4

As evident from the above table, the calculated number of electrons donated by one atom of the element is the same as the valency of that element and the position in the periodic table.

3. (2.11) **Resistivity of solid solution metal alloys: testing Nordheim's rule**

Nordheim's rule accounts for the increase in the resistivity from the scattering of electrons from the random distribution of impurity (solute) atoms in the host (solvent) crystal. Table 2.12 lists some solid solution metal alloys and gives the resistivity ρ at one composition X and asks for a prediction ρ' based on Nordheim's rule at another composition X'. Fill in the table for predicted ρ' and compare (i.e., compute the percent difference) the predicted values with the experimental values, and comment.

Table 2.12 Resistivities of some solid solution metal alloys



	Alloy							
	Ag–Au	**Au–Ag**	**Cu–Pd**	**Ag–Pd**	**Au–Pd**	**Pd–Pt**	**Pt–Pd**	**Cu–Ni**
X (at.%)	8.8% Au	8.77% Ag	6.2% Pd	10.1% Pd	8.88% Pd	7.66% Pt	7.1% Pd	2.16% Ni
ρ_o (nΩ m)	16.2	22.7	17	16.2	22.7	108	105.8	17
ρ at X (nΩ m)	44.2	54.1	70.8	59.8	54.1	188.2	146.8	50
C_{eff}								
X'	15.4% Au	24.4% Ag	13% Pd	15.2% Pd	17.1% Pd	15.5% Pt	13.8% Pd	23.4% Ni
ρ' at X' (nΩ m)								
ρ' at X' (nΩ m) Experimental	66.3	107.2	121.6	83.8	82.2	244	181	300

Table 2.12 NOTE: First symbol (e.g., Ag in AgAu) is the matrix (solvent) and the second (Au) is the added solute. X is in at.%, converted from traditional weight percentages reported with alloys. C_{eff} is the effective Nordheim coefficient in $\rho = \rho_0 + C_{eff} X(1-X)$.

Solution

Combined Matthiessen and Nordheim rule is

$$\rho_{\text{alloy}} = \rho_0 + C_{\text{eff}} X(1-X)$$

therefore, from the above equation effective Nordheim coefficient C_{eff} is

$$C_{\text{eff}} = \frac{\rho_{\text{alloy}} - \rho_0}{X(1-X)}$$

<u>Ag-Au:</u>

For this alloy, it is given that for X = 8.8% Au, ρ = 44.2 nΩ m, with ρ_0 = 16.2 nΩ m, the effective Nordheim coefficient C_{eff} is

$$C_{\text{eff}} = \frac{(44.2-16.2)\ \text{n}\Omega\,\text{m}}{0.088\times(1-0.088)} = 348.89\ \text{n}\Omega\ \text{m}$$

Now, for X'= 15.4% Au, the resistivity of the alloy will be

$$\rho' = 16.2\,\text{n}\Omega\,\text{m} + (348.88\,\text{n}\Omega\,\text{m})(0.154)(1-0.154) = 61.65\ \text{n}\Omega\ \text{m}$$

Similarly, the effective Nordheim coefficient C_{eff} and the resistivities of the alloys at X' are calculated for the various alloys and tabulated as follows,

Table 2Q11-1: Resistivities of solid solution metal alloys

	Alloy							
	Ag-Au	Au-Ag	Cu-Pd	Ag-Pd	Au-Pd	Pd-Pt	Pt-Pd	Cu-Ni

X (at.%)	8.8% Au	8.77% Ag	6.2% Pd	10.1% Pd	8.88% Pd	7.66% Pt	7.1% Pd	2.16% Ni
ρ_0 (nΩ m)	16.2	22.7	17	16.2	22.7	108	105.8	17
ρ at X (nΩ m)	44.2	54.1	70.8	59.8	54.1	188.2	146.8	50
C_{eff}	348.88	392.46	925.10	480.18	388.06	1133.85	621.60	1561.51
X'	15.4% Au	24.4% Ag	13% Pd	15.2% Pd	17.1% Pd	15.5% Pt	13.8% Pd	23.4% Ni
ρ' at X' (nΩ m)	61.65	95.09	121.63	78.09	77.71	256.51	179.74	296.89
ρ' at X' (nΩ m) Experimental	66.3	107.2	121.6	83.8	82.2	244	181	300
Percentage Difference	7.01% less	11.29% less	0.02% more	6.81% less	5.46% less	4.88% more	0.69% less	1.04% less

Comment: From the above table, the best case has a 0.02% difference and the worst case has a 11.3% difference. It is clear that the Nordheim rule can be very useful in predicting the approximate resistivity of a solid solution at one composition from the resistivity at a known composition.

4. (2.20) **Ag–Ni alloys (contact materials) and the mixture rules**

   Silver alloys, particularly Ag alloys with the precious metals Pt, Pd, Ni, and Au, are used extensively as contact materials in various switches. Table 2.15 shows the resistivities of four Ag–Ni alloys used in make-and-break as well as disconnect contacts with current ratings up to ~100 A.

   a. Ag–Ni is a two-phase alloy, a mixture of Ag-rich and Ni-rich phases. Using an appropriate mixture rule, predict the resistivity of the alloy and compare with the measured values in Table 2.15. Explain the difference between the predicted and experimental values.

   b. Compare the resistivity of Ag–10% Ni with that of Ag–10% Pd in Table 2.12. The resistivity of the Ag–Pd alloy is almost a factor of 3 greater. Ag–Pd is an isomorphous solid solution, whereas Ag–Ni is a two-phase mixture. Explain the difference in the resistivities of Ag–Ni and Ag–Pd.

Table 2.15 Resistivity of Ag-Ni contact alloys for switches

	Ni % in Ag-Ni alloy						
	0	10	15	20	30	40	100
d (g cm^{-3})	10.49	10.25	10.15	10.05	9.8	9.7	8.91
ρ (nΩ m)	16.9	18.7	19.0	20.0	24.4	27.0	71.0

NOTE: Compositions are in wt.%. Ag–10% Ni means 90% Ag–10% Ni. d = density and ρ = resistivity. Use volume fraction of Ni = $w_{\text{Ni}}(d_{\text{alloy}}/d_{\text{Ni}})$, where w_{Ni} is the Ni weight fraction, to convert wt.% to volume %. Data combined from various sources.

Solution

a. The Ni contents are given in wt.%. For volume fraction we use the relation

$$\chi_{Ni} = \frac{w_{Ni} d}{d_{Ni}}$$

where w_{Ni} is the weight fraction of Ni, d_{Ni} is the density of Ni and, d is the density of the alloy mixture. For example, for Ni-30% wt. the volume fraction of Ni in the alloy will be

$$\chi_{Ni} = \frac{(0.3)(9.8\times10^3\,\text{kg m}^{-3})}{(8.91\times10^3\,\text{kg m}^{-3})} = 0.33$$

First, we use Reynolds and Hough rule for mixture of dispersed phases to calculate the effective resistivity of the alloy. From Equation 2.28 we have

$$\frac{\sigma - \sigma_{\text{Ag}}}{\sigma + 2\sigma_{\text{Ag}}} = \chi_{\text{Ni}} \frac{\sigma_{\text{Ni}} - \sigma_{\text{Ag}}}{\sigma_{\text{Ni}} + 2\sigma_{\text{Ag}}}$$

Solving for Ni-30% wt., the R.H.S. of the above equation will be

$$= (0.33)\times\frac{(71.0\,\text{n}\Omega\,\text{m})^{-1} - (16.9\,\text{n}\Omega\,\text{m})^{-1}}{(71.0\,\text{n}\Omega\,\text{m})^{-1} + 2(16.9\,\text{n}\Omega\,\text{m})^{-1}} = -0.112\,(\text{n}\Omega\,\text{m})^{-1}$$

So that

$$\frac{\sigma - (16.9\,\text{n}\Omega\,\text{m})^{-1}}{\sigma + 2(16.9\,\text{n}\Omega\,\text{m})^{-1}} = -0.112\,(\text{n}\Omega\,\text{m})^{-1}$$

∴ $\sigma = 0.0412\ (\text{n}\Omega\ \text{m})^{-1}$

∴ $\rho = 24.25\ \text{n}\Omega\ \text{m}$.

Substitute the calculated value in the Reynolds and Hough rule as above, to find the effective resistivity of the alloy, which is $\rho = 24.25\ \text{n}\Omega\ \text{m}$. Similarly, the resistivity of alloy with other Ni contents is calculated and is tabulated below in Table 2Q20-1.

We can see that the Reynolds Hough rule provides a reasonable estimate for the alloy resistivity with the discrepancy being 7.5% at worst case.

Table 2Q20-1 Resistivity of Ag-Ni contact alloys for switches

Ni % in Ag-Ni	d (g cm^{-3})	χ_{Ni}	ρ_{eff} (Reynolds & Hough) (nΩ m)	Experimental (nΩ m)	Discrepancy (%)
10	10.25	0.115	19.06	18.7	1.93
15	10.15	0.171	20.24	19.0	6.53
20	10.05	0.226	21.50	20.0	7.50
30	9.8	0.330	24.25	24.4	0.62
40	9.7	0.435	27.59	27.0	2.20

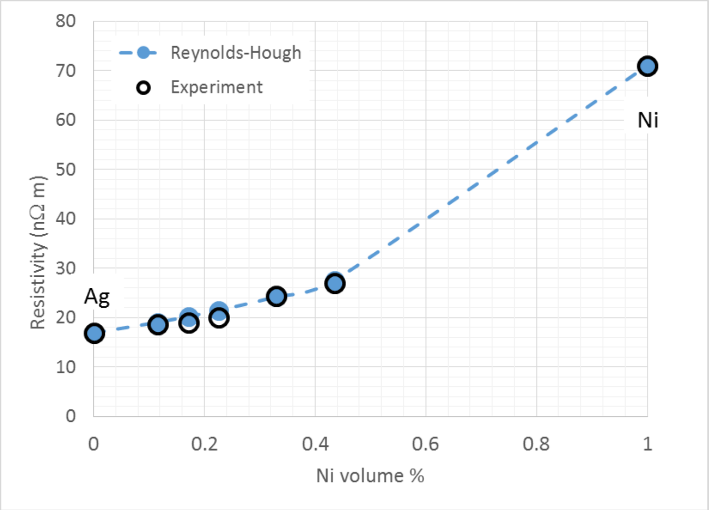


Figure 2Q20-1 Plot of effective and calculated resistivity for Ni-Ag alloys

b. 90%Ag-10% Ni, the solid is a <u>mixture</u>, and has two phases with an overall $\rho = 19.06$ nΩ m. On the other hand, for 90%Ag-10% Pd, the solid is a <u>solid solution</u> with $\rho = 59.8$ nΩ m, the value is roughly 3 times greater. The resistivity of a mixture is normally much lower than the resistivity of a *similar* solid solution. In a solid solution, the added impurities scatter electrons and increase the resistivity. In a mixture, each phase is almost like a "pure" metal, and the overall resistivity is simply an appropriate "averaging" or combination of the two resistivities.

Note: Data were extracted from http://www.electrical-contacts-wiki.com