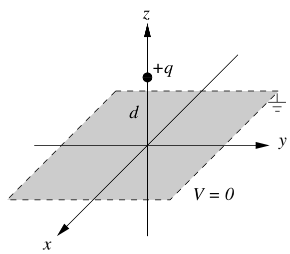
MSE 311
Fall 2021 Graugnard

Homework #1
Due 28 August 2021

---

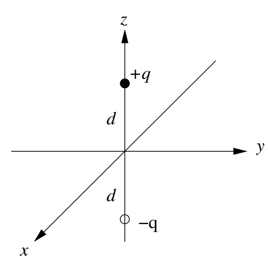
1 Consider a point charge held a distance d above an infinite grounded conducting plane, as shown below.



(a) Sketch the image charges and calculate the potential in the region above the plate.

(b) What is the force on the charge q?

Solution Mathematically, we need to solve Poisson's equation in the region of interest ($z > 0$). We can assemble a charge configuration that possesses the same boundary conditions as the grounded conducting plane ($V = 0$ in the xy-plane), if we place an image charge (same magnitude, opposite sign) at an equal distance on the other side of the xy-plane, as shown below.



For this configuration, we can simply write the potential for $z > 0$ as a superposition of the potential from each point charge.

$$\boxed{V(x, y, z) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{x^2 + y^2 + (z - d)^2}} + \frac{-q}{\sqrt{x^2 + y^2 + (z + d)^2}}\right]}$$

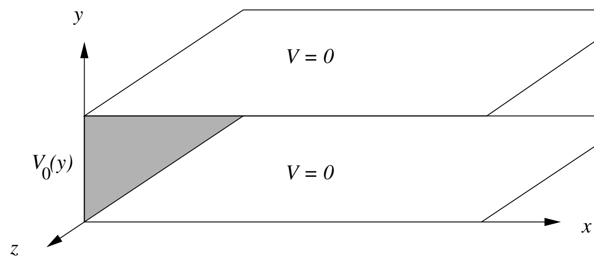
or

$$\boxed{\begin{aligned} V(\vec{r}) &= \frac{q}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{x^2+y^2+(z-d)^2}}\right. \\ &\left. -\frac{1}{\sqrt{x^2+y^2+(z+d)^2}}\right] \end{aligned}} \quad \text{(a)}$$

The force at the charge q can also be written down, but the vector nature must be taken into account.

$$\boxed{\vec{F}_q = \frac{1}{4\pi\epsilon_0}\frac{-q^2}{(2d)^2}\hat{z}} \quad (b)$$

1.2 Consider an infinitely long rectangular slot consisting of two infinite grounded plates that lie parallel to the xy plane. One plate is located at $y = 0$ and the other is at $y = a$. The bounded end of the slot consists of an isolated metal strip held at a potential $V_0(y)$. See the drawing below.



(a) Write down the form of Laplace's equation appropriate to this problem (being sure to explain any assumptions) and use separation of variables to determine the general solution for $V(x, y)$ inside the slot.

(b) Impose the boundary conditions appropriate at $x \to \infty$ and also at $y = 0$ and $y = a$ to simplify the solution.

(c) Now state the boundary condition at $x = 0$ and use the principle of superposition and this last boundary condition to determine the potential in the slot.

Solution This is a problem for separation of variables in cartesian coordinates. Start by assuming that the solution is independent of z and is separable, so $V(x, y) = X(x)Y(y)$. The solution for the potential must obey Laplace's equation within the slot, so we have

$$\nabla^2 V(x,y) = 0 \;\rightarrow\; \nabla^2[X(x)Y(y)] = Y(y)\frac{\partial^2 X(x)}{\partial x^2} + X(x)\frac{\partial^2 Y(y)}{\partial y^2} = 0.$$

Now, dividing by $V(x, y)$ yields

$$\frac{1}{X(x)}\frac{d^2X(x)}{dx^2} + \frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2} = 0,$$

which contains two terms of only a single variable. For this to be zero for any x and y, then each term must be a constant such that

$$\frac{1}{X(x)}\frac{d^2X(x)}{dx^2} = C_1 \quad \text{and} \quad \frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2} = C_2 \quad \text{where} \quad C_1 + C_2 = 0.$$

The boundary conditions for this problem are

$$\begin{aligned} V(x,0) &= 0 && (1)\\ V(x,a) &= 0 && (2)\\ V(x,y) &\rightarrow 0 \quad \text{as} \quad x \rightarrow \infty && (3)\\ V(0,y) &= V_0(y) && (4) \end{aligned}$$

Based on these boundary conditions, we know that the constant C_1 must be positive, while C_2 must be negative, so

$$C_1 = k^2 \quad \text{and} \quad C_2 = -k^2.$$

Thus we have two second order differential equations for $X(x)$ and $Y(y)$:

$$\frac{d^2X(x)}{dx^2} - k^2X(x) = 0 \quad \text{and} \quad \frac{d^2Y(y)}{dy^2} + k^2Y(y) = 0.$$

The solutions to these are

$$X(x) = Ae^{kx} + Be^{-kx} \quad \text{and} \quad Y(y) = C\sin(ky) + D\cos(ky).$$

Now we apply the boundary conditions: (1) requires $D = 0$, (2) requires $k = \frac{n\pi}{a}$, where n is an integer, and (3) requires $A = 0$. Thus, the general solution is (combining constants)

$$V(x, y) = \sum_{n=1}^{\infty} C_n e^{-n\pi x/a} \sin(n\pi y/a).$$

Applying the final boundary condition (4) yields

$$V(0, y) = \sum_{n} C_n \sin(n\pi y/a) = V_0(y).$$

Multiplying both sides by $sin(n'y)$ and integrating from 0 to a yields

$$C_n = \frac{2V_0}{n\pi}\big(1 - \cos(n\pi)\big) = \begin{cases} 0, & \text{if } n \text{ is even,} \\ \frac{4V_0}{n\pi}, & \text{if } n \text{ is odd.} \end{cases}$$

The only non-zero coefficients are for odd n, so the potential within the slot is

$$\boxed{V(x, y) = \frac{4V_0}{\pi} \sum_{n=1,3,5,\ldots} \frac{1}{n} e^{-n\pi x/a} \sin(n\pi y/a).}$$

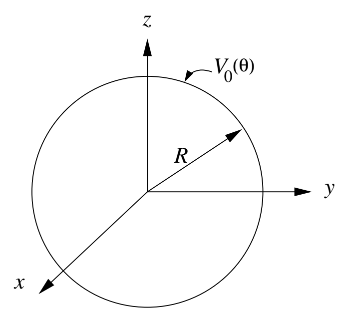
1.3 Consider a spherical shell of radius R for which the potential on the surface is given by $V(R,\theta) = V(\theta)$.

(a) Draw a sketch of the system, write down the form of Laplace's equation appropriate for this problem, and use separation of variables to give the *general* form of the solution for this system.

(b) We want to find the solution inside and outside the shell. Use the principle of superposition to write the form of the solutions that will remain finite, first for $r < R$, then for $r > R$.

(c) Now impose the boundary condition at $r = R$ and determine the potential both inside and outside the shell.

Solution **(a)** This is a problem for separation of variables in spherical coordinates. The problem has azimuthal symmetry, so the solution will be independent of ϕ.



Thus, we assume the solution has the form

$$V(r,\theta,\phi) = V(r,\theta) = R(r)\Theta(\theta).$$

Thus, we can insert this form of the potential into Laplace's equation and divide by $V(r,\theta)$ to obtain

$$\nabla^2 V(r,\theta) = \nabla^2[R(r)\Theta(\theta)] = 0$$
$$\Rightarrow \frac{1}{R(r)}\frac{d}{dr}\left(r^2\frac{dR(r)}{dr}\right) + \frac{1}{\sin\theta\;\Theta(\theta)}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta(\theta)}{d\theta}\right) = 0$$

Again, for this to be true for any r and θ, then each term must equal a constant. Here we'll let the constants be $\pm l(l+1)$. So we have 2 equations to solve:

$$\frac{d}{dr}\left(r^2\frac{dR(r)}{dr}\right) - l(l+1)R(r) = 0 \quad \Rightarrow \quad R(r) = A_l r^l + \frac{B_l}{r^{l+1}}$$
$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta(\theta)}{d\theta}\right) + l(l+1)\Theta(\theta) = 0 \quad \Rightarrow \quad \Theta(\theta) = P_l(\cos\theta)$$

$$\therefore \quad V(r,\theta) = \left(A_l r^l + \frac{B_l}{r^{l+1}}\right) P_l(\cos\theta)$$

(b) Now for $r < R$ in order for the potential to remain finite at the origin then $B_l = 0$. Likewise, for $r > R$ then $A_l = 0$. So

$$V(r,\theta) = \begin{cases} \sum_l A_l r^l P_l(\cos\theta) & (r \le R) \\ \sum_l \frac{B_l}{r^{l+1}} P_l(\cos\theta) & (r \ge R) \end{cases}$$

(c) At $r = R$, $V(r,\theta) = V_0(\theta)$, so we'll apply this to both equations at $r = R$ and use the orthogonality of the Legendre Polynomials to determine the non-zero coefficients.

$$\int_0^\pi V_0(\theta) P_m(\cos\theta)\sin\theta d\theta = \sum_l A_l R^l \underbrace{\int_0^\pi P_l(\cos\theta) P_m(\cos\theta)\sin\theta d\theta}_{\frac{2}{2m+1}\delta_{l,m}}$$

$$\therefore \quad A_m = \frac{2m+1}{2R^m}\int_0^\pi V_0(\theta) P_m(\cos\theta)\sin\theta d\theta$$

$$\int_0^\pi V_0(\theta) P_l(\cos\theta)\sin\theta d\theta = \sum_l \frac{B_l}{R^{l+1}} \underbrace{\int_0^\pi P_l(\cos\theta) P_m(\cos\theta)\sin\theta d\theta}_{\frac{2}{2m+1}\delta_{l,m}}$$

$$\therefore \quad B_m = \frac{2m+1}{2} R^{m+1}\int_0^\pi V_0(\theta) P_m(\cos\theta)\sin\theta d\theta$$

Now, to determine the coefficients A_m and B_m, we need to know the form of $V_0(\theta)$, which hopefully we could write in terms of Legendre Polynomials. Without $V_0(\theta)$, we cannot proceed further. See examples 3.6 and 3.7 in the text for more detail.
